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Associated Primes for Cohomology Modules

Jonathan Elmer

1. Introduction

Let k be a field of finite characteristic p , and G a finite group acting on the left on a finite dimensional k -vector space V . Then the dual vector space V^* is naturally a right kG -module, and the symmetric algebra of the dual, $R := \text{Sym}(V^*)$, is a polynomial ring over k on which G acts naturally by graded algebra automorphisms, and if k is algebraically closed can be regarded as the space $k[V]$ of polynomial functions on V . The G -fixed points of R under this action form a ring, which we denote by R^G and call the ring of invariants. If k is algebraically closed, R^G can be regarded as the set of G -invariant polynomial functions on V , or the ring of coordinate functions on the quotient space V/G . The ring of invariants R^G is the central object of study in invariant theory. The situation becomes modular when we assume p divides the order of G . Let P be a (fixed) Sylow- p -subgroup of G .

Since the ring of invariants R^G coincides with the zeroth cohomology $H^0(G, R)$, we can regard R^G as the zeroth degree part of the cohomology ring $H^*(G, R)$, and as such, the higher cohomology modules $H^i(G, R)$ become R^G -modules via the cup product. One can often learn more about the structure of modular rings of invariants by studying these higher cohomology modules; for example, in [3] Ellingsrud and Skjelbred showed that $H^i(G, R)$ is Cohen-Macaulay for G cyclic of order p . They then used this result to find a formula for the depth the ring of invariants R^G in this case. This approach was also used in [7], [9] and [10] to answer questions about the depth or Cohen-Macaulay property of modular invariant rings.

If $X < G$, we may define a mapping $\text{Tr}_X^G : R^X \rightarrow R^G$ as follows: let S be a set of right coset representatives of X in G . Then we define

$$\text{Tr}_X^G(x) := \sum_{g \in S} xg. \quad (1)$$

This mapping is often called the *relative transfer*, and induces mappings $\text{Tr}_X^G : H^i(X, R) \rightarrow H^i(G, R)$ also called the relative transfer. Both are surjective when the index of X in G is coprime to p . The image of the transfer map $\text{Tr}_X^G(R^X)$ is

an ideal in R^G called the *relative transfer ideal* which we denote by I_X^G . We may generalise this definition and define

$$I_\chi^G := \sum_{X \in \chi} I_X^G$$

for any set χ of subgroups of G . Relative transfer ideals and their radicals have been studied widely in connection with modular invariant theory. For example it is known that the quotient ring $R^G / \sqrt{I_{<P}^G}^1$ is always Cohen-Macaulay (see [6]).

Let $H^+(G, R)$ denote the set of positive degree elements of the cohomology ring $H^*(G, R)$, that is, we define $H^+(G, R) := \bigoplus_{i>0} H^i(G, R)$. The main purpose of this paper is to prove the following:

Theorem 1.1. *Let \mathfrak{p} be an associated prime ideal of the R^G -module $H^+(G, R)$. Then $\mathfrak{p} = \sqrt{I_\chi^G}$ for some set χ of subgroups of G .*

Remark: The relative transfer ideals I_χ^G defined above were first studied by Fleischmann ([5]), who proved the following formulae:

$$\sqrt{I_\chi^G} = \left(\bigcap_{X \in \chi'} ((g-1)V^* \mid g \in X)R \right) \cap R^G = \left(\bigcap_{X \in \chi'} \mathcal{I}(V^X) \right) \cap R^G \quad (2)$$

where $\chi' := \{Q \leq P \mid Q \not\leq X^g \text{ for any } g \in G \text{ and } X \in \chi\}$, and for a subspace W of V , $\mathcal{I}(W)$ denotes $\{f \in k[V] \mid f(W) = 0\}$. So we should be able to use these formulae along with Theorem 1.1 to construct some associated primes of cohomology modules.

Brief digression: Consider for a moment the cohomology ring $H := H^*(G, k)$ of a finite group G with coefficients in a field k whose characteristic divides the order of G . It is known (see, for example [2], Theorem 12.7.1) that the associated primes of the ring H take the form $\sqrt{\ker(\text{res}_E^G)}$ for certain elementary abelian subgroups E of G . Using a result of Benson ([1], Theorem 1.1), one can show this is equal to $\sqrt{\sum_{X \in \chi} \text{Tr}_X^G(H^*(X, k))}$ where χ is the set of subgroups of G not contained in any Sylow- p -subgroup of $C_G(E)$. So the associated primes of H are also radicals of relative transfer ideals. Whether this result and Theorem 1.1 are two examples of a more general phenomenon remains to be seen.

2. Annihilators in Cohomology

The following lemma is an observation of Lorenz and Pathak ([11], Lemma 1.3). It is a simple consequence of the transfer-restriction formula for cup products ([2], Theorem 4.4.2) and the starting point for our investigations. Throughout this section, let m be a strictly positive integer.

Lemma 2.1. *Suppose $\alpha \in H^m(G, R)$ satisfies $\text{res}_N^G(\alpha) = 0$. Then $\text{Ann}_{R^G}(\alpha) \geq I_N^G$.*

¹Here, “ $< P$ ” means the set of all proper subgroups of P .

Proof. Let $x \in R^N$. Then we have $\text{Tr}_N^G(x) \cdot \alpha = \text{Tr}_N^G(x \cdot \text{res}_N^G(\alpha)) = 0$. \square

Corollary 2.2. Suppose $\alpha \in H^m(G, R)$ and define

$$\chi(\alpha) := \{X \leq P \mid \text{res}_X^G(\alpha) = 0\}. \quad (3)$$

Then $\text{Ann}_{R^G}(\alpha) \geq I_{\chi(\alpha)}^G$.

Remark: Since the Sylow- p -subgroups of G are conjugate and $\text{Tr}_{X^g}^G(x) = \text{Tr}_X^G(xg)$, we gain nothing by considering the set of all subgroups $X \leq G$ on which $\text{res}_X^G(\alpha) = 0$.

The following result on annihilators is the key to proving our main theorem.

Lemma 2.3. Let $0 \neq \alpha \in H^m(G, R)$. Then we have

$$\text{Ann}_{R^G}(\alpha) \leq \sqrt{I_{\leq P}^G} = \mathcal{I}(V^P) \cap R^G.$$

Proof. The second statement is just (2) applied to the set $\{< P\}$ of all proper subgroups of P . The first is [7], Corollary 2.2, which is itself a consequence of a much more general result of Kemper ([10], Proposition 1.2). \square

Lemma 2.4. Let $0 \neq \alpha \in H^m(G, R)$. Then we have

$$\sqrt{I_{\chi(\alpha)}^G} = \bigcap_{X \in \chi'(\alpha)} (\mathcal{I}(V^X)) \cap R^G$$

where $\chi'(\alpha) := \{X \leq P \mid \text{res}_X^G(\alpha) \neq 0\}$.

Proof. Using (2), we must show that $\chi'(\alpha)$ as defined above is equal to

$$\{X \leq P \mid X \not\leq Y^g \text{ for any } g \in G \text{ and } Y \in \chi(\alpha)\}.$$

This is tantamount to proving that $\text{res}_X^G(\alpha) = 0$ implies $\text{res}_{X^g}^G(\alpha) = 0$ for all $g \in G$, which is well known and follows from the fact that conjugation map $(-)^{g^{-1}} : X^g \rightarrow X$ induces an isomorphism $i : H^*(X, R) \rightarrow H^*(X^g, R)$ satisfying $\text{res}_X^G = i \circ \text{res}_{X^g}^G$. \square

Our main theorem now follows from the following result:

Proposition 2.5. Suppose $\alpha \in H^m(G, R)$ and $\chi(\alpha)$ is defined as in Corollary 2.2. Then

$$\sqrt{\text{Ann}_{R^G}(\alpha)} = \sqrt{I_{\chi(\alpha)}^G}$$

Remark: Suppose $\alpha \in H^1(G, k)$. Then α can be viewed as a homomorphism from G to k , which has a well-defined kernel N . Kemper ([9], Proposition 3.4)² proved that $\sqrt{\text{Ann}_{R^G}(\alpha)} = \sqrt{I_N^G}$. For any subgroup $X \leq G$, we have $\text{res}_X^G(\alpha) = 0$ if and only if $X \leq N^g$ for some $g \in G$. So Proposition 2.5 may be viewed as a generalisation of this result.

²Kemper actually proved this result under the assumption that k is algebraically closed, although since Fleischmann's formulae (2) hold for an arbitrary field of characteristic p , the generalisation of his result to any field of characteristic p is easily obtained.

Proof. That $\sqrt{\text{Ann}_{R^G}(\alpha)} \geq \sqrt{I_{\chi(\alpha)}^G}$ is an immediate consequence of Corollary 2.2. To prove the reverse, let $y^n \in \text{Ann}_{R^G}(\alpha)$ for some $n \geq 0$. Let $Q \in \chi'(\alpha)$ and define $\beta := \text{res}_Q^G(\alpha) \neq 0$. Then we have

$$0 = y^n \cdot \alpha = \text{res}_Q^G(y^n \cdot \alpha) = y^n \cdot \beta$$

since $\text{res}_Q^G : H^*(G, R) \rightarrow H^*(Q, R)$ is a ring homomorphism which specialises to the inclusion $R^G \rightarrow R^Q$ on the degree zero part. This means that $y^n \in \text{Ann}_{R^Q}(\beta)$, so $y^n \in \mathcal{I}(V^Q) \cap R^Q$ by Lemma 2.3, and since this holds for every $Q \in \chi'(\alpha)$ we have

$$y^n \in R^G \cap \bigcap_{X \in \chi'(\alpha)} ((\mathcal{I}(V^X)) \cap R^X) = R^G \cap \bigcap_{X \in \chi'(\alpha)} (\mathcal{I}(V^X)) = \sqrt{I_{\chi(\alpha)}^G}$$

where the final equality follows from Lemma 2.4. Therefore $y \in \sqrt{I_{\chi(\alpha)}^G}$ as required. This completes the proof of Proposition 2.5, and since the associated primes of $H^+(G, R)$ are those annihilators of homogeneous $\alpha \in H^+(G, R)$ which are prime ideals, this completes the proof of Theorem 1.1 too. \square

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